JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 100 (1998) 145–160

Weak solvability of interior transmission problems via mixed finite elements and Dirichlet-to-Neumann mappings

Gabriel R. Barrenechea[a,1], Gabriel N. Gatica[a,*,1], George C. Hsiao[b]

[a] *Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile*

[b] *Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716, USA*

Received 31 March 1997

Abstract

We study the weak solvability of an interior linear-nonlinear transmission problem arising in steady heat transfer and potential theory. For the variational formulation, we use a Dirichlet-to-Neumann mapping on the interface, which is obtained from the application of the boundary integral method to the linear domain, and we utilize a mixed finite element method in the nonlinear region. Existence and uniqueness of solution for the continuous formulation are provided and general approximation results for a fully discrete Galerkin method are derived. In particular, a compatibility condition between the mesh sizes involved is deduced in order to conclude the solvability and stability of this Galerkin scheme.

Keywords: Transmission problem; Dirichlet-to-Neumann mapping; Boundary integral method; Mixed finite element; Non-conforming Galerkin scheme

1. Introduction

The combination of the finite element method with the boundary integral equation method has been extensively applied to solve several kinds of interior and exterior nonlinear-linear transmission problems. This includes the utilization of displacement-type finite elements and mixed finite elements, as well (see, e.g. [1, 6, 9–14], and the references therein).

Now, in the recent works [2, 8] we have combined a mixed finite element method with suitable Dirichlet-to-Neumann mappings to study the weak solvability and Galerkin approximations of *exterior* nonlinear-linear transmission problems in potential theory and elastostatics. It is worth remarking

* Corresponding author. E.mail: ggatica@ing-mat.udec.cl.

[1] The research of these authors was supported by FONDECYT-Chile through the project 1951143, by FONDAP-Conicyt through the Program A on Numerical Analysis and by the Dirección de Investigación, Universidad de Concepción, through the projects 97.013.012-1.0 and 97.013.010-1.0.

PII: S0377-0427(98)00185-X

that in both cases, and due to the exterior nature of the problems, one can choose a sufficiently large circle Γ as an additional interface boundary which allows to obtain an explicit formula for the Neumann data on Γ in terms of the corresponding Dirichlet data. More precisely, in [2] we used the Dirichlet-to-Neumann mapping from [9, 18] giving the normal derivative in terms only of the hypersingular boundary integral operator acting on the Dirichlet data. Similarly, in [8] we considered the Fourier series expansion from [16] expressing the tractions in terms of the displacements.

The purpose of this paper is to extend the joint applicability of mixed finite elements and Dirichlet-to-Neumann methods to solve *interior* linear-nonlinear transmission problems. To this end, we consider as a model the interior analogue of the problem studied in [2] which arises in steady heat transfer and potential theory. We note in this case that the resulting Dirichlet-to-Neumann mapping reduces to an implicit formula since the normal derivative on Γ appears as the solution of a boundary integral equation with the Dirichlet data on the right-hand side of it. The rest of our analysis follows very closely what we did in [2, 11–13], and hence we omit several details in some of the proofs. It is important to observe that, because of the implicit Dirichlet-to-Neumann mapping, the usual Galerkin scheme must be replaced by a non-conforming one which depends on two finite element subspaces. A compatibility condition between the corresponding mesh sizes is needed to guarantee the solvability and stability of this modified discrete scheme.

The rest of the paper is presented as follows. In Section 2 we describe our interior linear-nonlinear transmission problem and reduce it to an equivalent nonlocal boundary value problem. The weak formulation and the corresponding results on existence and uniqueness of solution are given in Section 3. Finally, in Section 4 we introduce the Galerkin schemes, study their solvability and provide the error estimates.

2. The transmission problem

Let Ω_0 be a bounded simply connected region in $\boldsymbol{R}^2$ with Lipschitz continuous boundary Γ. Also, let Ω be the annular region bounded by Γ and another Lipschitz closed curve Γ_1 whose interior contains $\overline{\Omega}_0$. Further, let Γ_{D} and Γ_{N} be two disjoint subsets of Γ_1 such that $\Gamma_1 = \bar{\Gamma}_{\mathrm{D}} \cup \bar{\Gamma}_{\mathrm{N}}$. In addition, let $a_i : \Omega \times \boldsymbol{R}^2 \rightarrow \boldsymbol{R}$, $i = 1, 2$ be nonlinear mappings satisfying the same regularity assumptions specified in [2] (Carathéodory condition, growth condition, strong monotonicity and Lipschitz-continuity). Then, given $(f, g) \in L^2(\Omega) \times H^{-1/2}(\Gamma_{\mathrm{N}})$, we consider the following transmission problem: *Find* $(u_0, u) \in H^1(\Omega_0) \times H^1(\Omega)$ *such that*

$$\begin{aligned}
&\operatorname{div} \boldsymbol{\sigma}_0 = 0 \quad \text{in} \quad \Omega_0,\\
&u_0 = u \quad \text{and} \quad \boldsymbol{\sigma}_0 \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \boldsymbol{v} \quad \text{on} \quad \Gamma,\\
&-\operatorname{div} \boldsymbol{\sigma} = f \quad \text{in} \quad \Omega,\\
&u = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}} \quad \text{and} \quad \boldsymbol{\sigma} \cdot \boldsymbol{n} = g \quad \text{on} \quad \Gamma_{\mathrm{N}},
\end{aligned} \tag{2.1}$$

where $\boldsymbol{v}$ and $\boldsymbol{n}$ denote the unit outward normals to $\partial\Omega_0$ and $\partial\Omega$ (note that $\boldsymbol{n} = -\boldsymbol{v}$ on Γ), respectively, and $\boldsymbol{\sigma}_0$, $\boldsymbol{\sigma}$ are given by

$$\boldsymbol{\sigma}_0 = \nabla u_0 \quad \text{in} \quad \Omega_0, \qquad \boldsymbol{\sigma} = \begin{bmatrix} a_1(\cdot, \nabla u) \\ \\ a_2(\cdot, \nabla u) \end{bmatrix} \quad \text{in} \quad \Omega.$$

In order to derive the Dirichlet-to-Neumann mapping on Γ, we first apply the boundary integral equation method in Ω_0. In fact, using the Green representation formula for the harmonic function u_0, the jump properties of the boundary potentials and the transmission conditions from (2.1), we get the identity

$$V(\boldsymbol{\sigma}\cdot v) = \left(\tfrac{1}{2}\boldsymbol{I} + \boldsymbol{K}\right)\left(u\big|_{\Gamma}\right) \quad \text{on } \Gamma. \tag{2.2}$$

Here, $\boldsymbol{V}$ and $\boldsymbol{K}$ denote the boundary integral operators associated with the simple and double layer potentials, respectively, which are defined by

$$(\boldsymbol{V}\lambda)(x) := \int_{\Gamma} E(x,y)\,\lambda(y)\,\mathrm{d}s_y \quad \forall x\in\Gamma,\ \forall\lambda\in H^{-1/2}(\Gamma),$$

$$(\boldsymbol{K}\lambda)(x) := \int_{\Gamma} \left\{\frac{\partial}{\partial v_y}E(x,y)\right\}\lambda(y)\,\mathrm{d}s_y \quad \forall x\in\Gamma,\ \forall\lambda\in H^{1/2}(\Gamma),$$

where $E(x,y) := 1/2\pi\log\{1/(|x-y|)\}$ is the fundamental solution of the Laplacian.

The main properties of these operators are stated in the following lemma (see [5, 17]).

Lemma 2.1. *For a boundary Γ of class $C^{0,1}$, and for each $\delta\in[-\frac{1}{2},\frac{1}{2}]$, the operators $\boldsymbol{V}: H^{-1/2+\delta}(\Gamma) \to H^{1/2+\delta}(\Gamma)$ and $\boldsymbol{K}: H^{1/2+\delta}(\Gamma)\to H^{1/2+\delta}(\Gamma)$ are continuous. In addition, if* $\operatorname{diam}(\Omega_0) < 1$, *$\boldsymbol{V}$ is bijective.*

Hence, throughout the rest of the paper we assume that Γ is of class $C^{0,1}$ and that $\operatorname{diam}(\Omega_0) < 1$.

According to the previous lemma, we can define the continuous linear operator $\Lambda: H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$,

$$\Lambda\lambda := \boldsymbol{V}^{-1}\left(\tfrac{1}{2}\boldsymbol{I} + \boldsymbol{K}\right)\lambda \quad \forall\lambda\in H^{1/2}(\Gamma),$$

which is also continuous from $H^{1/2+\delta}(\Gamma)$ into $H^{-1/2+\delta}(\Gamma)$ for all $\delta\in[-\frac{1}{2},\frac{1}{2}]$. In addition, it is not difficult to prove (cf. [4]) that

$$\langle\lambda, \Lambda\lambda\rangle \geqslant 0 \quad \forall\lambda\in H^{1/2}(\Gamma), \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. It follows from (2.2) that our Neumann data $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ may be represented as

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = -\Lambda\left(u\big|_{\Gamma}\right) \quad \text{on } \Gamma,$$

which becomes the announced Dirichlet-to-Neumann mapping.

On the other hand, we now introduce the auxiliary unknowns $\boldsymbol{t} := \nabla u$ in Ω, $\xi := u\big|_{\Gamma}$ on Γ, and set the notation

$$\boldsymbol{a}(\cdot,\boldsymbol{t}) := \begin{bmatrix} a_1(\cdot,\boldsymbol{t}) \\ a_2(\cdot,\boldsymbol{t}) \end{bmatrix} \quad \text{in } \Omega.$$

In this way, the original transmission problem can be rewritten as the following nonlocal boundary value problem: *Find* $(\boldsymbol{t}, \boldsymbol{\sigma}, u, \xi)$ *such that*

$$\begin{aligned}
&\boldsymbol{\sigma} \cdot \boldsymbol{n} = -\Lambda\xi \quad \text{on } \Gamma, \\
&\boldsymbol{t} = \nabla u \quad \text{in } \Omega, \\
&\boldsymbol{\sigma} = \boldsymbol{a}(\cdot, \boldsymbol{t}) \quad \text{in } \Omega, \\
&-\operatorname{div} \boldsymbol{\sigma} = f \quad \text{in } \Omega, \\
&\boldsymbol{\sigma} \cdot \boldsymbol{n} = g \quad \text{on} \quad \Gamma_{\mathrm{N}}, \; u = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}.
\end{aligned} \tag{2.4}$$

3. The weak formulation and the operator equation

We first recall that $H^{-1/2}(\Gamma_{\mathrm{N}})$ denotes the dual of $\tilde{H}^{1/2}(\Gamma_{\mathrm{N}})$, which is defined as the interpolation space

$$\tilde{H}^{1/2}(\Gamma_{\mathrm{N}}) := [H_0^1(\Gamma_{\mathrm{N}}), L^2(\Gamma_{\mathrm{N}})]_{1/2},$$

(see [19, p. 10]), where $H_0^1(\Gamma_{\mathrm{N}})$ is the closure of $C_0^\infty(\Gamma_{\mathrm{N}})$ in $H^1(\Gamma_{\mathrm{N}})$. We notice that $H^{-1/2}(\Gamma_{\mathrm{N}})$ is essentially distinct of $\tilde{H}^{-1/2}(\Gamma_{\mathrm{N}})$, which is the dual of $H^{1/2}(\Gamma_{\mathrm{N}})$ (the usual space of restrictions of the elements of $H^{1/2}(\Gamma_1)$ to Γ_{N}).

Now, for the derivation of the weak formulation we multiply the second equation in (2.4) by $\boldsymbol{\tau} \in H(\operatorname{div}, \Omega)$ and integrate by parts in Ω to obtain

$$-\int_\Omega \boldsymbol{t} \cdot \boldsymbol{\tau} \, \mathrm{d}x - \int_\Omega u \operatorname{div} \boldsymbol{\tau} \, \mathrm{d}x - \langle \eta, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\Gamma_{\mathrm{N}}} + \langle \xi, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle = 0, \tag{3.1}$$

where $\eta := -\, u\big|_{\Gamma_{\mathrm{N}}} \in \tilde{H}^{1/2}(\Gamma_{\mathrm{N}})$ is a further unknown and $\langle \cdot, \cdot \rangle_{\Gamma_{\mathrm{N}}}$ denotes the duality pairing on $\tilde{H}^{1/2}(\Gamma_{\mathrm{N}}) \times H^{-1/2}(\Gamma_{\mathrm{N}})$. Next, we define the space

$$K := \{\boldsymbol{\tau} \in H(\operatorname{div}, \Omega) \,|\, \operatorname{div} \boldsymbol{\tau} = 0 \quad \text{in} \quad \Omega \text{ and } \boldsymbol{\tau} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}} \cup \Gamma\},$$

and observe from (3.1) that $\boldsymbol{t} \in H_1 := [L^2(\Omega)]^2/K$. Hence, testing the third equation in (2.4) against $\boldsymbol{s} \in H_1$ we get

$$\int_\Omega \boldsymbol{a}(\cdot, \boldsymbol{t}) \cdot \boldsymbol{s} \, \mathrm{d}x - \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{s} \, \mathrm{d}x = 0 \quad \forall \boldsymbol{s} \in H_1. \tag{3.2}$$

From (2.4) and (3.2) it follows that $\boldsymbol{\sigma} \in H_2 := H(\operatorname{div}, \Omega)/K$. Finally, the remaining equations in (2.4) are tested against $v \in L^2(\Omega)$, $\lambda \in H^{1/2}(\Gamma)$ and $\psi \in \tilde{H}^{1/2}(\Gamma_{\mathrm{N}})$, respectively, which gives

$$-\int_\Omega v \operatorname{div} \boldsymbol{\sigma} \, \mathrm{d}x = \int_\Omega f \, v \, \mathrm{d}x \quad \forall v \in L^2(\Omega), \tag{3.3}$$

$$-\langle \lambda, \boldsymbol{\sigma} \cdot \boldsymbol{n} \rangle - \langle \lambda, \Lambda \xi \rangle = 0 \qquad \forall \lambda \in H^{1/2}(\Gamma), \tag{3.4}$$

and

$$\langle \psi, \boldsymbol{\sigma} \cdot \boldsymbol{n} \rangle_{\Gamma_{\mathrm{N}}} = \langle \psi, g \rangle_{\Gamma_{\mathrm{N}}} \qquad \forall \psi \in \tilde{H}^{1/2}(\Gamma_{\mathrm{N}}). \tag{3.5}$$

Collecting (3.1)–(3.5), we see that the weak formulation of (2.4) reads: *Find* $(\boldsymbol{t},\boldsymbol{\sigma},\xi,u,\eta)\in H_1\times H_2\times H^{1/2}(\Gamma)\times L^2(\Omega)\times \tilde{H}^{1/2}(\Gamma_{\mathrm{N}})$ such that:

$$\begin{aligned}
&-\int_\Omega \boldsymbol{t}\cdot\boldsymbol{\tau}\,\mathrm{d}x-\int_\Omega u\,\mathrm{div}\boldsymbol{\tau}\,\mathrm{d}x-\langle\eta,\boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_{\Gamma_{\mathrm{N}}}+\langle\xi,\boldsymbol{\tau}\cdot\boldsymbol{n}\rangle=0,\\
&\int_\Omega \boldsymbol{a}(\cdot,\boldsymbol{t})\cdot\boldsymbol{s}\,\mathrm{d}x-\int_\Omega\boldsymbol{\sigma}\cdot\boldsymbol{s}\,\mathrm{d}x=0,\\
&-\int_\Omega v\,\mathrm{div}\,\boldsymbol{\sigma}\,\mathrm{d}x=\int_\Omega f\,v\,\mathrm{d}x,\\
&-\langle\lambda,\boldsymbol{\sigma}\cdot\boldsymbol{n}\rangle-\langle\lambda,\Lambda\,\xi\rangle=0,\\
&-\langle\psi,\boldsymbol{\sigma}\cdot\boldsymbol{n}\rangle_{\Gamma_{\mathrm{N}}}=-\langle\psi,g\rangle_{\Gamma_{\mathrm{N}}},
\end{aligned}\tag{3.6}$$

for all $(\boldsymbol{s},\boldsymbol{\tau},\lambda,v,\psi)\in H_1\times H_2\times H^{1/2}(\Gamma)\times L^2(\Omega)\times\tilde{H}^{1/2}(\Gamma_{\mathrm{N}})$.

The analysis of the existence and uniqueness of solutions is easier if we first reduce this variational formulation to an equivalent operator equation. For this purpose, we denote $\boldsymbol{X}:=H_1,\boldsymbol{Y}:=H_2\times H^{1/2}(\Gamma)\times L^2(\Omega)\times\tilde{H}^{1/2}(\Gamma_{\mathrm{N}})$, and define the continuous operators

$$\boldsymbol{T}:\boldsymbol{X}\to\boldsymbol{X}',\quad \boldsymbol{Q}:\boldsymbol{Y}\to\boldsymbol{X}',\quad \boldsymbol{Q}':\boldsymbol{X}\to\boldsymbol{Y}',\quad \boldsymbol{S}:\boldsymbol{Y}\to\boldsymbol{Y}',$$

as follows

$$\begin{aligned}
&[\boldsymbol{T}(\boldsymbol{t}),\boldsymbol{s}]_{X'\times X}:=\int_\Omega \boldsymbol{a}(\cdot,\boldsymbol{t})\cdot\boldsymbol{s}\,\mathrm{d}x,\\
&[\boldsymbol{Q}(\boldsymbol{\sigma},\xi,u,\eta),\boldsymbol{s}]_{X'\times X}:=-\int_\Omega\boldsymbol{\sigma}\cdot\boldsymbol{s}\,\mathrm{d}x,\\
&[\boldsymbol{Q}'(\boldsymbol{t}),(\boldsymbol{\tau},\lambda,v,\psi)]_{Y'\times Y}:=-\int_\Omega\boldsymbol{t}\cdot\boldsymbol{\tau}\,\mathrm{d}x,\\
&[\boldsymbol{S}(\boldsymbol{\sigma},\xi,u,\eta),(\boldsymbol{\tau},\lambda,v,\psi)]_{Y'\times Y}:=\int_\Omega u\,\mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}x-\langle\xi,\boldsymbol{\tau}\cdot\boldsymbol{n}\rangle+\langle\eta,\boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_{\Gamma_{\mathrm{N}}}\\
&\qquad\qquad+\langle\lambda,\boldsymbol{\sigma}\cdot\boldsymbol{n}\rangle+\langle\lambda,\Lambda\xi\rangle+\int_\Omega v\,\mathrm{div}\,\boldsymbol{\sigma}\,\mathrm{d}x+\langle\psi,\boldsymbol{\sigma}\cdot\boldsymbol{n}\rangle_{\Gamma_{\mathrm{N}}}.
\end{aligned}$$

Hence, the weak formulation (3.6) can be rewritten as the following operator equation: *Find* $(\boldsymbol{t},(\boldsymbol{\sigma},\xi,u,\eta))\in\boldsymbol{X}\times\boldsymbol{Y}$ *such that*

$$\begin{bmatrix}\boldsymbol{T} & \boldsymbol{Q}\\ \boldsymbol{Q}' & -\boldsymbol{S}\end{bmatrix}\begin{bmatrix}\boldsymbol{t}\\ (\boldsymbol{\sigma},\xi,u,\eta)\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{F}\end{bmatrix},\tag{P}$$

where $\boldsymbol{0}\in\boldsymbol{X}'$ is the null functional and $\boldsymbol{F}\in\boldsymbol{Y}'$ is defined by

$$[\boldsymbol{F},(\boldsymbol{\tau},\lambda,v,\psi)]_{Y'\times Y}:=\int_\Omega fv\,\mathrm{d}x-\langle\psi,g\rangle_{\Gamma_{\mathrm{N}}}.$$

We remark that the structure of $(\boldsymbol{P})$ is the same as that obtained in [2, 11–14].

The solvability of $(\boldsymbol{P})$ is based on the following result.

Lemma 3.1. *Assume that $\boldsymbol{S}$ is continuous and bijective, and let $\mathscr{T}: \boldsymbol{X} \to \boldsymbol{X}'$ be the nonlinear operator given by $\mathscr{T} := \boldsymbol{T} + \boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{Q}'$. Then the matrix operator $\begin{bmatrix} \boldsymbol{T} & \boldsymbol{Q} \\ \boldsymbol{Q}' & -\boldsymbol{S} \end{bmatrix}$ is bijective if and only if $\mathscr{T}$ is bijective.*

Proof. See [6, 11]. □

In order to apply this lemma we need to prove the bijectivity of $\boldsymbol{S}$. To do this, we first see that $\boldsymbol{S}$ may be written as an operator arising from a variational problem with constraints. Indeed, let us denote $\mathscr{X} := H_2 \times H^{1/2}(\Gamma)$, $\mathscr{M} := L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_N)$, and define the bounded bilinear forms $\boldsymbol{A}: \mathscr{X} \times \mathscr{X} \to \boldsymbol{R}$, $\boldsymbol{B}: \mathscr{X} \times \mathscr{M} \to \boldsymbol{R}$, as follows

$$\boldsymbol{A}((\boldsymbol{\sigma}, \xi), (\boldsymbol{\tau}, \lambda)) := -\langle \xi, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle + \langle \lambda, \boldsymbol{\sigma} \cdot \boldsymbol{n} \rangle + \langle \lambda, \Lambda \xi \rangle,$$

$$\boldsymbol{B}((\boldsymbol{\sigma}, \xi), (v, \psi)) := \int_\Omega v \operatorname{div} \boldsymbol{\sigma} \, \mathrm{d}x + \langle \psi, \boldsymbol{\sigma} \cdot \boldsymbol{n} \rangle_{\Gamma_N}.$$

It follows that $\boldsymbol{S}$ may be written in the following way:

$$\begin{aligned}[\boldsymbol{S}(\boldsymbol{\sigma}, \xi, u, \eta), (\boldsymbol{\tau}, \lambda, v, \psi)]_{Y' \times Y} = {} & \boldsymbol{A}((\boldsymbol{\sigma}, \xi), (\boldsymbol{\tau}, \lambda)) \\ & + \boldsymbol{B}((\boldsymbol{\tau}, \lambda), (u, \eta)) + \boldsymbol{B}((\boldsymbol{\sigma}, \xi), (v, \psi)).\end{aligned}$$

Therefore, in order to conclude the bijectivity of $\boldsymbol{S}$, and based in Brezzi's theory for variational problems with constraints, it suffices to show that $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the usual inf–sup conditions. These results are collected in the following lemmata.

Lemma 3.2. *Let $\mathscr{V}$ be the kernel of the operator induced by $\boldsymbol{B}$, that is*

$$\mathscr{V} := \{ (\boldsymbol{\tau}, \lambda) \in \mathscr{X} \mid \boldsymbol{B}((\boldsymbol{\tau}, \lambda), (v, \psi)) = 0 \ \ \forall \, (v, \psi) \in \mathscr{M} \}.$$

Then there exists $\alpha > 0$ such that

$$\sup_{\substack{(\boldsymbol{\tau}, \lambda) \in V \\ (\boldsymbol{\tau}, \lambda) \neq 0}} \frac{\boldsymbol{A}((\boldsymbol{\sigma}, \xi), (\boldsymbol{\tau}, \lambda))}{\| (\boldsymbol{\tau}, \lambda) \|_{\mathscr{X}}} \geqslant \alpha \, \| (\boldsymbol{\sigma}, \xi) \|_{\mathscr{X}}, \qquad \forall (\boldsymbol{\sigma}, \xi) \in V,$$

and

$$\sup_{\substack{(\boldsymbol{\sigma}, \xi) \in V \\ (\boldsymbol{\sigma}, \xi) \neq 0}} \frac{\boldsymbol{A}((\boldsymbol{\sigma}, \xi), (\boldsymbol{\tau}, \lambda))}{\| (\boldsymbol{\sigma}, \xi) \|_{\mathscr{X}}} \geqslant \alpha \, \| (\boldsymbol{\tau}, \lambda) \|_{\mathscr{X}}, \qquad \forall (\boldsymbol{\tau}, \lambda) \in V.$$

Proof. It can be found, with minor changes, in [2] (Lemma 4.2). □

The inf–sup condition for $\boldsymbol{B}$ is stated in the following lemma.

Lemma 3.3. *There exists $\beta > 0$ such that*

$$\sup_{\substack{(\tau,\lambda)\in\mathcal{X}\\(\tau,\lambda)\neq 0}} \frac{\boldsymbol{B}((\tau,\lambda),(v,\psi))}{\|(\tau,\lambda)\|_{\mathcal{X}}} \geqslant \beta\ \|(v,\psi)\|_{\mathcal{M}} \qquad \forall\,(v,\psi)\in\mathcal{M}.$$

Proof. See Lemma 2.6 [11]. □

As a consequence of the previous lemmata and the usual Babuska–Brezzi theory for constrained variational problems (see, e.g. [3, Chap. II, Theorem 1.1]), we conclude that the operator $\boldsymbol{S}$ is bijective and has a bounded inverse $\boldsymbol{S}^{-1}$.

Hence, the operator equation $(\boldsymbol{P})$ is equivalent to the following formulation: *Find $\boldsymbol{t}\in\boldsymbol{X}$ such that*

$$\mathcal{T}(\boldsymbol{t}) = \boldsymbol{Q}\,\boldsymbol{S}^{-1}\,\boldsymbol{F}. \tag{3.7}$$

Once $\boldsymbol{t}$ is found from (3.7), we compute

$$(\sigma,\xi,u,\eta) = \boldsymbol{S}^{-1}\,[\boldsymbol{Q}'(\boldsymbol{t}) - \boldsymbol{F}].$$

We are now in a position to establish the solvability of $(\boldsymbol{P})$ (equivalently (3.7)).

Theorem 3.4. *There exists a unique $\boldsymbol{t}\in\boldsymbol{X}$ such that $\mathcal{T}(\boldsymbol{t})=\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{F}$. Moreover, the operator equation $(\boldsymbol{P})$ has a unique solution, which is given by $(\boldsymbol{t},(\sigma,\xi,u,\eta))\in\boldsymbol{X}\times\boldsymbol{Y}$, with $(\sigma,\xi,u,\eta)=\boldsymbol{S}^{-1}\,[\boldsymbol{Q}'(\boldsymbol{t})-\boldsymbol{F}]$.*

Proof. We note first that the assumptions on the nonlinear coefficientes a_i allow us to prove that the operator $\boldsymbol{T}$ is strongly monotone and Lipschitz-continuous (see Theorem 4.6 in [2] for details).

On the other hand, let us define the mapping $\hat{\boldsymbol{I}}: \boldsymbol{Y}\to\boldsymbol{Y}$ by $\hat{\boldsymbol{I}}(\tau,\lambda,v,\psi) := (\tau,\lambda,-v,-\psi)$ for all $(\tau,\lambda,v,\psi)\in\boldsymbol{Y}$. It is easy to see that $\boldsymbol{Q}\hat{\boldsymbol{I}} = \boldsymbol{Q}$. In addition, using (2.3) we observe that for all $(\tau,\lambda,v,\psi)\in\boldsymbol{Y}$,

$$[\boldsymbol{S}(\tau,\lambda,v,\psi),\hat{\boldsymbol{I}}(\tau,\lambda,v,\psi)]_{Y'\times Y} = \langle\lambda,\Lambda\lambda\rangle \geqslant 0.$$

Thus, for all $\boldsymbol{s}\in\boldsymbol{X}$ we have

$$\begin{aligned}[\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{Q}'\,\boldsymbol{s},\boldsymbol{s}]_{X'\times X} &= [\boldsymbol{Q}\hat{\boldsymbol{I}}(\boldsymbol{S}^{-1}\boldsymbol{Q}'\boldsymbol{s}),\boldsymbol{s}]_{X'\times X} = [\boldsymbol{Q}'\boldsymbol{s},\hat{\boldsymbol{I}}(\boldsymbol{S}^{-1}\boldsymbol{Q}'\boldsymbol{s})]_{Y'\times Y}\\ &= [\boldsymbol{S}(\boldsymbol{S}^{-1}\boldsymbol{Q}'\boldsymbol{s}),\hat{\boldsymbol{I}}(\boldsymbol{S}^{-1}\boldsymbol{Q}'\boldsymbol{s})]_{Y'\times Y} \geqslant 0,\end{aligned}$$

which shows that $\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{Q}'$ is positive semi-definite.

Hence, since $\mathcal{T} = \boldsymbol{T} + \boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{Q}'$, we deduce that $\mathcal{T}$ is also strongly monotone and Lipschitz-continuous. Then, by applying a classical result from nonlinear functional analysis (see, e.g. [20, Theorem 3.3.23]) we conclude the bijectivity of $\mathcal{T}$. This finishes the proof. □

4. The Galerkin approximations

We consider finite-dimensional subspaces $\boldsymbol{X}_h, \mathcal{X}_h$ and $\mathcal{M}_h$ of $\boldsymbol{X}$, $\mathcal{X}$ and $\mathcal{M}$, respectively, so that $\boldsymbol{Y}_h := \mathcal{X}_h \times \mathcal{M}_h$ becomes the corresponding subspace of $\boldsymbol{Y}$. For each $(\boldsymbol{s},(\tau,\lambda,v,\psi))\in\boldsymbol{X}\times\boldsymbol{Y}$

we denote dist $(s, \boldsymbol{X}_h)$ and dist$((\tau, \lambda, v, \psi), \boldsymbol{Y}_h)$ the distances to the subspaces $\boldsymbol{X}_h$ and $\boldsymbol{Y}_h$, respectively. We also introduce the canonical injections $\boldsymbol{i}_h : \boldsymbol{X}_h \hookrightarrow \boldsymbol{X}$ and $\boldsymbol{j}_h : \boldsymbol{Y}_h \hookrightarrow \boldsymbol{Y}$, with adjoints $\boldsymbol{i}'_h : \boldsymbol{X}' \hookrightarrow \boldsymbol{X}'_h$ and $\boldsymbol{j}'_h : \boldsymbol{Y}' \hookrightarrow \boldsymbol{Y}'_h$, respectively. Then the Galerkin scheme associated with $(\boldsymbol{P})$ reads: *Find* $(\boldsymbol{t}_h, (\boldsymbol{\sigma}_h, \xi_h, u_h, \eta_h)) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h$ *such that*

$$\begin{bmatrix} \boldsymbol{T}_h & \boldsymbol{Q}_h \\ \boldsymbol{Q}'_h & -\boldsymbol{S}_h \end{bmatrix} \begin{bmatrix} \boldsymbol{t}_h \\ (\boldsymbol{\sigma}_h, \xi_h, u_h, \eta_h) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{F}_h \end{bmatrix}, \tag{$\boldsymbol{P}$}_h$$

where $\boldsymbol{F}_h := \boldsymbol{j}'_h \boldsymbol{F}$, $\boldsymbol{T}_h := \boldsymbol{i}'_h \boldsymbol{T} \boldsymbol{i}_h : \boldsymbol{X}_h \to \boldsymbol{X}'_h$, $\boldsymbol{Q}_h := \boldsymbol{i}'_h \boldsymbol{Q} \boldsymbol{j}_h : \boldsymbol{Y}_h \to \boldsymbol{X}'_h$ and $\boldsymbol{S}_h := \boldsymbol{j}'_h \boldsymbol{S} \boldsymbol{j}_h : \boldsymbol{Y}_h \to \boldsymbol{Y}'_h$. Note that the operator $\boldsymbol{S}_h$ can be written as

$$\begin{aligned}[][\boldsymbol{S}_h(\boldsymbol{\sigma}_h, \xi_h, u_h, \eta_h), (\boldsymbol{\tau}_h, \lambda_h, v_h, \psi_h)]_{\boldsymbol{Y}'_h \times \boldsymbol{Y}_h} &= \boldsymbol{A}((\boldsymbol{\sigma}_h, \xi_h), (\boldsymbol{\tau}_h, \lambda_h)) \\ &\quad + \boldsymbol{B}((\boldsymbol{\tau}_h, \lambda_h), (u_h, \eta_h)) + \boldsymbol{B}((\boldsymbol{\sigma}_h, \xi_h), (v_h, \psi_h)). \end{aligned}$$

We now state the invertibility of $\boldsymbol{S}_h$.

Theorem 4.1. *Assume that there exist* $\tilde{\alpha}, \tilde{\beta} > 0$, *independent of* h, *such that* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *satisfy the discrete inf-sup conditions on* $\mathscr{V}_h$ *and on* $\mathscr{X}_h \times \mathscr{M}_h$, *respectively, where*

$$\mathscr{V}_h := \{(\boldsymbol{\tau}_h, \lambda_h) \in \mathscr{X}_h \mid \boldsymbol{B}((\boldsymbol{\tau}_h, \lambda_h), (v_h, \psi_h)) = 0 \quad \forall (v_h, \psi_h) \in \mathscr{M}_h\}.$$

Then for any $\boldsymbol{G}_h \in \boldsymbol{Y}'_h := \mathscr{X}'_h \times \mathscr{M}'_h$, *there exists a unique* $\boldsymbol{S}_h^{-1}(\boldsymbol{G}_h) \in \boldsymbol{Y}_h$. *Furthermore, there exist positive constants* C_1, C_2, *independent of* h, *such that*

$$\| \boldsymbol{j}_h \boldsymbol{S}_h^{-1}(\boldsymbol{G}_h) \|_{\boldsymbol{Y}} \leqslant C_1 \, \| \boldsymbol{G}_h \|_{\boldsymbol{Y}'_h} \qquad \forall \boldsymbol{G}_h \in \boldsymbol{Y}'_h,$$

and

$$\| \boldsymbol{S}^{-1}(\boldsymbol{G}) - \boldsymbol{j}_h \boldsymbol{S}_h^{-1} \boldsymbol{j}'_h(\boldsymbol{G}) \|_{\boldsymbol{Y}} \leqslant C_2 \operatorname{dist}(\boldsymbol{S}^{-1}(\boldsymbol{G}), \boldsymbol{Y}_h) \quad \forall \boldsymbol{G} \in \boldsymbol{Y}'.$$

Proof. It suffices to apply the standards results due to Brezzi (see, e.g. [15, Chap. II, Theorem 1.1] or [3, Chap. II, Theorem 2.1]). □

In this way we can define the operator $\mathscr{T}_h : \boldsymbol{X}_h \to \boldsymbol{X}'_h$ by

$$\mathscr{T}_h := \boldsymbol{T}_h + \boldsymbol{Q}_h \boldsymbol{S}_h^{-1} \boldsymbol{Q}'_h,$$

and find that the Galerkin scheme $(\boldsymbol{P})_h$ is equivalent to the following formulation: *Find* $\boldsymbol{t}_h \in \boldsymbol{X}_h$ *such that* $\mathscr{T}_h(\boldsymbol{t}_h) = \boldsymbol{Q}_h \boldsymbol{S}_h^{-1} \boldsymbol{F}_h$, and then compute

$$(\boldsymbol{\sigma}_h, \xi_h, u_h, \eta_h) = \boldsymbol{S}_h^{-1}[\boldsymbol{Q}'_h(\boldsymbol{t}_h) - \boldsymbol{F}_h].$$

Now, similarly as in Theorem 3.4 one can prove that $\mathscr{T}_h$ is also strongly monotone and Lipschitz continuous on $\boldsymbol{X}_h$, and hence existence, uniqueness and approximation results for $(\boldsymbol{P})_h$ can be obtained.

Theorem 4.2. *Assume the same hypothesis of Theorem* 4.1. *Then, the Galerkin scheme* $(\boldsymbol{P})_h$ *has a unique solution* $(\boldsymbol{t}_h, (\boldsymbol{\sigma}_h, \xi_h, u_h, \eta_h)) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h$. *Moreover, there exists* $C > 0$, *independent of* h, *such that*

$$\begin{aligned} &\| \boldsymbol{t} - \boldsymbol{t}_h \|_X + \| (\boldsymbol{\sigma}, \xi, u, \eta) - (\boldsymbol{\sigma}_h, \xi_h, u_h, \eta_h) \|_Y \\ &\leqslant C \ \{\operatorname{dist}(\boldsymbol{t}, \boldsymbol{X}_h) + \operatorname{dist}((\boldsymbol{\sigma}, \xi, u, \eta), \boldsymbol{Y}_h)\} . \end{aligned} \tag{4.1}$$

Proof. It remains to prove the Cea estimate (4.1). This is identical to the proof of the Theorem 5.2 in [2] and follows the same arguments shown below in Theorem 4.5. We omit further details at this stage. □

At this point we must recall that the Galerkin scheme $(\boldsymbol{P})_h$ involves the mapping Λ which requires the inversion of the single layer potential $\boldsymbol{V}$. Hence, from the practical point of view, this scheme is not suitable for computations. In order to overcome this difficulty, in what follows we introduce a nonconforming Galerkin method for $(\boldsymbol{P})$ which basically consists of replacing Λ by an appropriate boundary element approximation (for a similar scheme connected to a finite element approximation, see [7]).

We first let $H_h(\Gamma)$ and $H_{\hat{h}}(\Gamma)$ be finite element subspaces of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively, and from now on we assume that $H_h(\Gamma)$ is the second component of $\mathscr{X}_h$. Also, we suppose that these spaces satisfy the following approximation properties and inverse assumption:

$(\boldsymbol{AP})_h$ for all $0 \leqslant l \leqslant t \leqslant 1$ and for all $\lambda \in H^t(\Gamma)$

$$\inf_{\lambda_h \in H_h(\Gamma)} \| \lambda - \lambda_h \|_{H^l(\Gamma)} \leqslant C h^{t-l} \| \lambda \|_{H^t(\Gamma)},$$

$(\boldsymbol{AP})_{\hat{h}}$ for all $-1 \leqslant m \leqslant s \leqslant 0$ and for all $\lambda \in H^s(\Gamma)$

$$\inf_{\lambda_{\hat{h}} \in H_{\hat{h}}(\Gamma)} \| \lambda - \lambda_{\hat{h}} \|_{H^m(\Gamma)} \leqslant C \hat{h}^{s-m} \| \lambda \|_{H^s(\Gamma)},$$

$(\boldsymbol{IA})_h$ for all $0 \leqslant p \leqslant q \leqslant 1$ and for all $\lambda_h \in H_h(\Gamma) \cap H^q(\Gamma)$

$$\| \lambda_h \|_{H^q(\Gamma)} \leqslant C h^{p-q} \| \lambda_h \|_{H^p(\Gamma)} .$$

Furthermore, we assume that there exists $\delta > 0$ such that $H_h(\Gamma) \subseteq H^{1/2+\delta}(\Gamma)$ and $H_{\hat{h}}(\Gamma) \subseteq H^{-1/2+\delta}(\Gamma)$. Let us note that the simplest choice of elements (piecewise linear for $H_h(\Gamma)$ and piecewise constants for $H_{\hat{h}}(\Gamma)$) satisfies the above with $\delta = \frac{1}{2}$.

Now, given $\xi_h \in H_h(\Gamma)$, we let $\chi_{\hat{h}} \in H_{\hat{h}}(\Gamma)$ be the unique solution of

$$\langle \boldsymbol{V} \chi_{\hat{h}}, \lambda_{\hat{h}} \rangle = \langle (\tfrac{1}{2} \boldsymbol{I} + \boldsymbol{K}) \, \xi_h, \lambda_{\hat{h}} \rangle \qquad \forall \lambda_{\hat{h}} \in H_{\hat{h}}(\Gamma).$$

This induces the definition of the operator

$$\begin{aligned} &\Lambda_{\hat{h}} \, : \, H_h(\Gamma) \longrightarrow H_{\hat{h}}(\Gamma) \\ &\xi_h \longrightarrow \Lambda_{\hat{h}} \xi_h := \chi_{\hat{h}} , \end{aligned}$$

which constitutes a boundary element approximation of $\Lambda \xi_h$. Hence, using the results in [17] we deduce the uniform boundedness of the operators $\Lambda_{\hat{h}}$ and the classical Cea estimate

$$\| \Lambda \xi_h - \Lambda_{\hat{h}} \xi_h \|_{H^{-1/2}(\Gamma)} \leqslant C \operatorname{dist}(\Lambda \xi_h, H_{\hat{h}}(\Gamma)) \qquad \forall \xi_h \in H_h(\Gamma). \tag{4.2}$$

With this, we replace $\boldsymbol{S}_h$ by the bounded operator $\boldsymbol{S}_{\hat{h}}: \boldsymbol{Y}_h \to \boldsymbol{Y}'_h$,

$$[\boldsymbol{S}_{\hat{h}}(\sigma_h, \xi_h, u_h, \eta_h), (\tau_h, \lambda_h, v_h, \psi_h)]_{\boldsymbol{Y}'_h \times \boldsymbol{Y}_h} := \boldsymbol{A}_{\hat{h}}((\sigma_h, \xi_h), (\tau_h, \lambda_h)) \\ + \boldsymbol{B}((\tau_h, \lambda_h), (u_h, \eta_h)) + \boldsymbol{B}((\sigma_h, \xi_h), (v_h, \psi_h)),$$

where $\boldsymbol{A}_{\hat{h}}: \mathscr{X}_h \times \mathscr{X}_h \to \boldsymbol{R}$ is the bounded bilinear form obtained from $\boldsymbol{A}$ after replacing Λ by $\Lambda_{\hat{h}}$, that is

$$\boldsymbol{A}_{\hat{h}}((\sigma_h, \xi_h), (\tau_h, \lambda_h)) := -\langle \xi_h, \tau_h \cdot \boldsymbol{n} \rangle + \langle \lambda_h, \sigma_h \cdot \boldsymbol{n} \rangle + \langle \lambda_h, \Lambda_{\hat{h}} \xi_h \rangle.$$

Hence, our nonconforming Galerkin scheme is: *Find* $(\boldsymbol{t}_{h,\hat{h}}, (\sigma_{h,\hat{h}}, \xi_{h,\hat{h}}, u_{h,\hat{h}}, \eta_{h,\hat{h}})) \in \boldsymbol{X}_h \times \boldsymbol{Y}_h$ *such that*:

$$\begin{bmatrix} \boldsymbol{T}_h & \boldsymbol{Q}_h \\ \boldsymbol{Q}'_h & -\boldsymbol{S}_{\hat{h}} \end{bmatrix} \begin{bmatrix} \boldsymbol{t}_{h,\hat{h}} \\ (\sigma_{h,\hat{h}}, \xi_{h,\hat{h}}, u_{h,\hat{h}}, \eta_{h,\hat{h}}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{F}_h \end{bmatrix}, \qquad (\boldsymbol{P})_{h,\hat{h}}$$

Similarly as for $(\boldsymbol{P})$ and $(\boldsymbol{P})_h$, we now need to guarantee the invertibility of $\boldsymbol{S}_{\hat{h}}$. We have the following previous result.

Lemma 4.3. *Let us suppose that $\boldsymbol{A}$ satisfies the discrete inf-sup condition uniformly on $\mathscr{V}_h$. Then there exists $C_0 > 0$ such that for all $h > 0$ and for all $\hat{h} \leqslant C_0 h$, $\boldsymbol{A}_{\hat{h}}$ satisfies also the discrete inf–sup condition uniformly on $\mathscr{V}_h$.*

Proof. Let $\tilde{\alpha} > 0$ be the constant for the discrete inf–sup condition of $\boldsymbol{A}$ on $\mathscr{V}_h$. It follows that for all $(\sigma_h, \xi_h) \in V_h$,

$$\begin{aligned} \tilde{\alpha} \| (\sigma_h, \xi_h) \|_{\mathscr{X}} &\leqslant \sup_{\substack{(\tau_h, \lambda_h) \in V_h \\ (\tau_h, \lambda_h) \neq 0}} \frac{\boldsymbol{A}((\sigma_h, \xi_h), (\tau_h, \lambda_h))}{\| (\tau_h, \lambda_h) \|_{\mathscr{X}}} \\ &\leqslant \sup_{\substack{(\tau_h, \lambda_h) \in \mathscr{V}_h \\ (\tau_h, \lambda_h) \neq 0}} \frac{|\boldsymbol{A}((\sigma_h, \xi_h), (\tau_h, \lambda_h)) - \boldsymbol{A}_{\hat{h}}((\sigma_h, \xi_h), (\tau_h, \lambda_h))|}{\| (\tau_h, \lambda_h) \|_{\mathscr{X}}} \\ &\quad + \sup_{\substack{(\tau_h, \lambda_h) \in \mathscr{V}_h \\ (\tau_h, \lambda_h) \neq 0}} \frac{\boldsymbol{A}_{\hat{h}}((\sigma_h, \xi_h), (\tau_h, \lambda_h))}{\| (\tau_h, \lambda_h) \|_{\mathscr{X}}}. \end{aligned} \tag{4.3}$$

Then, using the definitions of $\boldsymbol{A}$ and $\boldsymbol{A}_{\hat{h}}$, (4.2), $(\boldsymbol{AP})_{\hat{h}}$ and $(\boldsymbol{IA})_h$, we can write

$$\begin{aligned} &\sup_{\substack{(\tau_h, \lambda_h) \in V_h \\ (\tau_h, \lambda_h) \neq 0}} \frac{|\boldsymbol{A}((\sigma_h, \xi_h), (\tau_h, \lambda_h)) - \boldsymbol{A}_{\hat{h}}((\sigma_h, \xi_h), (\tau_h, \lambda_h))|}{\| (\tau_h, \lambda_h) \|_{\mathscr{X}}} \\ &= \sup_{\substack{(\tau_h, \lambda_h) \in V_h \\ (\tau_h, \lambda_h) \neq 0}} \frac{|\langle \lambda_h, \Lambda \xi_h - \Lambda_{\hat{h}} \xi_h \rangle|}{\| (\tau_h, \lambda_h) \|_{\mathscr{X}}} \end{aligned}$$

$$\leqslant \sup_{\substack{(\boldsymbol{\tau}_h,\lambda_h)\in V_h\\(\boldsymbol{\tau}_h,\lambda_h)\neq 0}} \left\{ \frac{\|\lambda_h\|}{\|(\tau_h,\lambda_h)\|} \operatorname{dist}(\Lambda\xi_h, H_{\hat h}(\Gamma)) \right\} \leqslant C\,\hat h^{\delta}\ \|\Lambda\xi_h\|_{H^{-1/2+\delta}(\Gamma)}$$

$$\leqslant C\,\hat h^{\delta}\ \|\xi_h\|_{H^{1/2+\delta}(\Gamma)} \leqslant C \left[\frac{\hat h}{h}\right]^{\delta} \|\xi_h\|_{H^{1/2}(\Gamma)}$$

$$\leqslant C \left[\frac{\hat h}{h}\right]^{\delta} \|(\boldsymbol{\sigma}_h,\xi_h)\|_{\mathscr{X}}\ .$$

With this, (4.3) becomes

$$\left\{\tilde\alpha - C\left[\frac{\hat h}{h}\right]^{\delta}\right\} \|(\boldsymbol{\sigma}_h,\xi_h)\|_{\mathscr{X}} \leqslant \sup_{\substack{(\boldsymbol{\tau}_h,\lambda_h)\in \mathscr{X}_h\\(\boldsymbol{\tau}_h,\lambda_h)\neq 0}} \frac{\boldsymbol{A}_{\hat h}((\boldsymbol{\sigma}_h,\xi_h),(\tau_h,\lambda_h))}{\|(\tau_h,\lambda_h)\|_{\mathscr{X}}},$$

and the proof is finished by choosing $C_0>0$ such that $\tilde\alpha - C\,C_0^{\delta} > 0$. $\square$

According to the previous lemma, we conclude that under the same hypotheses of the Theorem 4.1 the operator $\boldsymbol{S}_{\hat h}$ is bijective and has a uniformly bounded inverse $\boldsymbol{S}_{\hat h}^{-1}$ for all $\hat h \leqslant C_0\,h$. Consequently, we can define the operator $\mathscr{T}_{\hat h} := \boldsymbol{T}_h + \boldsymbol{Q}_h \boldsymbol{S}_{\hat h}^{-1}\boldsymbol{Q}_h'$ and notice that the nonconforming Galerkin scheme $(\boldsymbol{P})_{h,\hat h}$ is equivalent to the following formulation: *Find* $\boldsymbol{t}_{h,\hat h}\in \boldsymbol{X}_h$ *such that*

$$\mathscr{T}_{\hat h}(\boldsymbol{t}_{h,\hat h}) = \boldsymbol{Q}_h \boldsymbol{S}_{\hat h}^{-1}\boldsymbol{F}_h\,,$$

and then compute

$$(\boldsymbol{\sigma}_{h,\hat h},\xi_{h,\hat h},u_{h,\hat h},\eta_{h,\hat h}) = \boldsymbol{S}_{\hat h}^{-1}[\boldsymbol{Q}_h'(\boldsymbol{t}_{h,\hat h}) - \boldsymbol{F}_h]\,. \tag{4.4}$$

We are ready now to establish the solvability of $(\boldsymbol{P})_{h,\hat h}$.

Theorem 4.4. *Assume the same hypothesis of Theorem* 4.1. *Then, there exists* $\tilde C_0>0$ *such that for all* $h>0$ *and for all* $\hat h\leqslant \tilde C_0\,h$, *the nonconforming Galerkin scheme* $(\boldsymbol{P})_{h,\hat h}$ *has a unique solution* $(\boldsymbol{t}_{h,\hat h},(\boldsymbol{\sigma}_{h,\hat h},\xi_{h,\hat h},u_{h,\hat h},\eta_{h,\hat h}))\in \boldsymbol{X}_h\times \boldsymbol{Y}_h$.

Proof. It suffices to show that $\mathscr{T}_{\hat h}$ is strongly monotone and Lipschitz-continuous. In fact, we first note that $\boldsymbol{T}_h$ is strongly monotone and Lipschitz continuous on $\boldsymbol{X}_h$. In particular, there exists $C>0$, independent of h, such that for all $\boldsymbol{s}_h,\ \boldsymbol{t}_h\in \boldsymbol{X}_h$,

$$[\boldsymbol{T}_h(\boldsymbol{s}_h) - \boldsymbol{T}_h(\boldsymbol{t}_h), \boldsymbol{s}_h - \boldsymbol{t}_h]_{X_h'\times X_h} \geqslant C\ \|\boldsymbol{s}_h - \boldsymbol{t}_h\|_X^2\ . \tag{4.5}$$

On the other hand, it is easy to see that $\boldsymbol{Q}_h\hat{\boldsymbol{I}} = \boldsymbol{Q}_h$, where $\hat{\boldsymbol{I}}$ is the linear operator introduced in Theorem 3.4. Then, using the definition of $\boldsymbol{S}_{\hat h}$, (2.3) and the approximation properties of $\Lambda_{\hat h}$, we conclude that

$$
\begin{aligned}
&[\boldsymbol{S}_{\hat h}(\tau_h,\lambda_h,v_h,\psi_h),\hat{\boldsymbol{I}}(\tau_h,\lambda_h,v_h,\psi_h)]_{Y'_h\times Y_h}=\langle\lambda_h,\Lambda_{\hat h}\lambda_h\rangle\\
&=\langle\lambda_h,\Lambda\,\lambda_h\rangle-\langle\lambda_h,\Lambda\,\lambda_h-\Lambda_{\hat h}\lambda_h\rangle\geqslant -C\left[\frac{\hat h}{h}\right]^{\delta}\|\lambda_h\|^2_{H^{1/2}(\Gamma)}\\
&\geqslant -C\left[\frac{\hat h}{h}\right]^{\delta}\|(\tau_h,\lambda_h,v_h,\psi_h)\|^2_Y\,.
\end{aligned}
$$

Having this in mind, it follows that

$$
\begin{aligned}
[\boldsymbol{Q}_h\boldsymbol{S}_{\hat h}^{-1}\boldsymbol{Q}'_h s_h,s_h]_{X'_h\times X_h}&=[\boldsymbol{Q}_h\hat{\boldsymbol{I}}(\boldsymbol{S}_{\hat h}^{-1}\boldsymbol{Q}'_h s_h),s_h]_{X'_h\times X_h}\\
&=[\boldsymbol{S}_{\hat h}(\boldsymbol{S}_{\hat h}^{-1}\boldsymbol{Q}'_h s_h),\hat{\boldsymbol{I}}(\boldsymbol{S}_{\hat h}^{-1}\boldsymbol{Q}'_h s_h)]_{Y'_h\times Y_h}\\
&\geqslant -C\left[\frac{\hat h}{h}\right]^{\delta}\|\boldsymbol{S}_{\hat h}^{-1}\boldsymbol{Q}'_h s_h\|^2_Y\geqslant -\overline{C}\left[\frac{\hat h}{h}\right]^{\delta}\|s_h\|^2_X,
\end{aligned}
\tag{4.6}
$$

where, in the last inequality, we have used the uniform boundedness of $\boldsymbol{S}_{\hat h}^{-1}$ and the fact that $\|\boldsymbol{Q}'_h\|\leqslant 1$. Therefore, by virtue of (4.5) and (4.6), we obtain

$$
\begin{aligned}
[\mathscr{T}_{\hat h}(s_h)-\mathscr{T}_{\hat h}(t_h),s_h-t_h]_{X'_h\times X_h}&=[\boldsymbol{T}_h(s_h)-\boldsymbol{T}_h(t_h),s_h-t_h]_{X'_h\times X_h}\\
&\quad+[\boldsymbol{Q}_h\boldsymbol{S}_{\hat h}^{-1}\boldsymbol{Q}'_h(s_h-t_h),s_h-t_h]_{X'_h\times X_h}\\
&\geqslant\left\{C-\overline{C}\left[\frac{\hat h}{h}\right]^{\delta}\right\}\|s_h-t_h\|^2_X,
\end{aligned}
$$

and hence, $\tilde C_0$ is chosen such that $C-\overline{C}\tilde C_0^{\delta}>0$ and $\tilde C_0\leqslant C_0$, where C_0 is the constant given by Lemma 4.3. Finally, the Lipschitz-continuity of $\mathscr{T}_{\hat h}$ follows from the same property of $\boldsymbol{T}_h$ and from the uniform boundedness of the linear operator $\boldsymbol{Q}_h\boldsymbol{S}_{\hat h}^{-1}\boldsymbol{Q}'_h$. □

We end this section with the following Cea-type estimate for the nonconforming Galerkin solution.

Theorem 4.5. *Let $\tilde C_0>0$ be the constant from Theorem 4.4 and let $(\boldsymbol{t},(\boldsymbol{\sigma},\xi,u,\eta))\in\boldsymbol{X}\times\boldsymbol{Y}$ and $(\boldsymbol{t}_{h,\hat h},(\boldsymbol{\sigma}_{h,\hat h},\xi_{h,\hat h},u_{h,\hat h},\eta_{h,\hat h}))\in\boldsymbol{X}_h\times\boldsymbol{Y}_h$ be the unique solutions of $(\boldsymbol{P})$ and $(\boldsymbol{P})_{h,\hat h}$, respectively. Then, there exists a constant $C>0$, independent of h and $\hat h$, such that for all $h>0$ and for all $\hat h\leqslant\tilde C_0 h$, the following error estimate holds:*

$$
\begin{aligned}
&\|\boldsymbol{t}-\boldsymbol{t}_{h,\hat h}\|_X+\|(\boldsymbol{\sigma},\xi,u,\eta)-(\boldsymbol{\sigma}_{h,\hat h},\xi_{h,\hat h},u_{h,\hat h},\eta_{h,\hat h})\|_Y\\
&\quad\leqslant C\,\{\operatorname{dist}(\boldsymbol{t},\boldsymbol{X}_h)+\operatorname{dist}((\boldsymbol{\sigma},\xi,u,\eta),\boldsymbol{Y}_h)+\operatorname{dist}(\Lambda\xi,H_{\hat h}(\Gamma))\}\,.
\end{aligned}
$$

Proof. We first observe that the computation of $(\boldsymbol{\sigma}_{h,\hat h},\xi_{h,\hat h},u_{h,\hat h},\eta_{h,\hat h})$ (cf. (4.4)) reduces to the solution of a nonconforming Galerkin scheme for a linear variational problem with constraints. Hence, by using the Strang-type estimate from ([3], Chap. 2, Proposition 2.16), we conclude that there exists a constant $C > 0$ such that

$$
\begin{aligned}
&\| (\boldsymbol{\sigma}, \xi, u, \eta) - (\boldsymbol{\sigma}_{h,\hat{h}}, \xi_{h,\hat{h}}, u_{h,\hat{h}}, \eta_{h,\hat{h}}) \|_Y \\
&\leqslant C \left\{ \operatorname{dist}((\boldsymbol{\sigma}, \xi), \mathscr{X}_h) + \operatorname{dist}((u, \eta), \mathscr{M}_h) \right. \\
&\quad + \sup_{\substack{(\boldsymbol{\tau}_h, \lambda_h) \in V_h \\ (\boldsymbol{\tau}_h, \lambda_h) \neq 0}} \frac{|A((\boldsymbol{\sigma}, \xi), (\boldsymbol{\tau}_h, \lambda_h)) - A_{\hat{h}}((\boldsymbol{\sigma}, \xi), (\boldsymbol{\tau}_h, \lambda_h))|}{\| (\boldsymbol{\tau}_h, \lambda_h) \|_{\mathscr{X}}} \\
&\quad \left. + \sup_{\substack{(\boldsymbol{\tau}_h, \lambda_h, v_h, \psi_h) \in V_h \times \mathscr{M}_h \\ (\boldsymbol{\tau}_h, \lambda_h, v_h, \psi_h) \neq 0}} \frac{\left| [\boldsymbol{Q}_h'(\boldsymbol{t}_{h,\hat{h}}) - \boldsymbol{Q}'(\boldsymbol{t}), (\boldsymbol{\tau}_h, \lambda_h, v_h, \psi_h)]_{Y_h' \times Y_h} \right|}{\| (\boldsymbol{\tau}_h, \lambda_h, v_h, \psi_h) \|_Y} \right\} \\
&\leqslant C \{ \operatorname{dist}((\boldsymbol{\sigma}, \xi, u, \eta), \boldsymbol{Y}_h) + \sup_{\substack{(\boldsymbol{\tau}_h, \lambda_h) \in V_h \\ (\boldsymbol{\tau}_h, \lambda_h) \neq 0}} \frac{|\langle \lambda_h, \Lambda\xi - \Lambda_{\hat{h}}\xi \rangle|}{\| (\boldsymbol{\tau}_h, \lambda_h) \|_{\mathscr{X}}} + \| \boldsymbol{t} - \boldsymbol{t}_{h,\hat{h}} \|_X \} \\
&\leqslant C \{ \operatorname{dist}((\boldsymbol{\sigma}, \xi, u, \eta), \boldsymbol{Y}_h) + \operatorname{dist}(\Lambda\xi, H_{\hat{h}}(\Gamma)) + \| \boldsymbol{t} - \boldsymbol{t}_{h,\hat{h}} \|_X \}.
\end{aligned} \tag{4.7}
$$

Hence, it remains to estimate the error $\| \boldsymbol{t} - \boldsymbol{t}_{h,\hat{h}} \|_X$. For this purpose, we now introduce the so-called quasi-Galerkin approximation $\boldsymbol{t}_h^* \in \boldsymbol{X}_h$ defined as the unique solution of

$$[\mathscr{T}(\boldsymbol{t}_h^*), \boldsymbol{s}_h]_{X' \times X} = [\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{F}, \boldsymbol{s}_h]_{X' \times X} \qquad \forall \boldsymbol{s}_h \in \boldsymbol{X}_h,$$

or, in an operator form

$$(\boldsymbol{i}_h' \mathscr{T} \boldsymbol{i}_h)\boldsymbol{t}_h^* = \boldsymbol{i}_h'(\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{F}).$$

It is not difficult to see, using the properties of $\mathscr{T}$, that

$$\| \boldsymbol{t} - \boldsymbol{t}_h^* \|_X \leqslant C \operatorname{dist}(\boldsymbol{t}, \boldsymbol{X}_h). \tag{4.8}$$

Therefore, it suffices to bound the error $\| \boldsymbol{t}_h^* - \boldsymbol{t}_{h,\hat{h}} \|_X$. From the uniform strong monotonicity of $\mathscr{T}_{\hat{h}}$, we see that

$$
\begin{aligned}
C \| \boldsymbol{t}_h^* - \boldsymbol{t}_{h,\hat{h}} \|_X^2 &\leqslant [\mathscr{T}_{\hat{h}}(\boldsymbol{t}_h^*) - \mathscr{T}_{\hat{h}}(\boldsymbol{t}_{h,\hat{h}}), \boldsymbol{t}_h^* - \boldsymbol{t}_{h,\hat{h}}]_{X_h' \times X_h} \\
&= [\boldsymbol{T}_h(\boldsymbol{t}_h^*), \boldsymbol{t}_h^* - \boldsymbol{t}_{h,\hat{h}}]_{X_h' \times X_h} + [\boldsymbol{Q}_h \boldsymbol{S}_{\hat{h}}^{-1} \boldsymbol{Q}_h' \boldsymbol{t}_h^*, \boldsymbol{t}_h^* - \boldsymbol{t}_{h,\hat{h}}]_{X_h' \times X_h} \\
&\quad - [\boldsymbol{Q}_h \boldsymbol{S}_{\hat{h}}^{-1} \boldsymbol{F}_h, \boldsymbol{t}_h^* - \boldsymbol{t}_{h,\hat{h}}]_{X_h' \times X_h}.
\end{aligned} \tag{4.9}
$$

Now, from the definition of $\boldsymbol{t}_h^*$ it follows that

$$[\boldsymbol{T}_h(\boldsymbol{t}_h^*),\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}=[(\boldsymbol{i}_h'\mathscr{T}\boldsymbol{i}_h)\boldsymbol{t}_h^*-\boldsymbol{i}_h'(\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{Q}')\boldsymbol{i}_h\boldsymbol{t}_h^*,\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}$$
$$=[\boldsymbol{i}_h'(\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{F}),\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}-[\boldsymbol{i}_h'(\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{Q}')\boldsymbol{i}_h\boldsymbol{t}_h^*,\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}. \quad (4.10)$$

Replacing (4.10) back into (4.9) we get

$$C\;\|\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}\|_X^2$$
$$\leqslant[\boldsymbol{i}_h'(\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{F}),\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}-[\boldsymbol{i}_h'(\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{Q}')\boldsymbol{i}_h\boldsymbol{t}_h^*,\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}$$
$$+[\boldsymbol{Q}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{Q}_h'\boldsymbol{t}_h^*,\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}-[\boldsymbol{Q}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{F}_h,\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}$$
$$=[(\boldsymbol{Q}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{Q}_h'-\boldsymbol{i}_h'(\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{Q}')\boldsymbol{i}_h)\boldsymbol{t}_h^*,\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}$$
$$+[\boldsymbol{i}_h'(\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{F})-\boldsymbol{Q}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{F}_h,\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}.$$

Using that $\boldsymbol{Q}_h:=\boldsymbol{i}_h'\boldsymbol{Q}\boldsymbol{j}_h$ and that $\boldsymbol{Q}_h':=\boldsymbol{j}_h'\boldsymbol{Q}'\boldsymbol{i}_h$, and adding and substracting $\boldsymbol{Q}'\boldsymbol{t}$, we find that

$$[(\boldsymbol{Q}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{Q}_h'-\boldsymbol{i}_h'(\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{Q}')\boldsymbol{i}_h)\boldsymbol{t}_h^*,\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}$$
$$=[\boldsymbol{Q}(\boldsymbol{j}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{Q}_h'-\boldsymbol{S}^{-1}\boldsymbol{Q}'\boldsymbol{i}_h)\boldsymbol{t}_h^*,\boldsymbol{i}_h(\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}})]_{X'\times X}$$
$$=[\boldsymbol{Q}'\boldsymbol{i}_h(\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}),(\boldsymbol{j}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{j}_h'-\boldsymbol{S}^{-1})\boldsymbol{Q}'(\boldsymbol{i}_h\boldsymbol{t}_h^*-\boldsymbol{t})]_{Y'\times Y}$$
$$+[\boldsymbol{Q}'\boldsymbol{i}_h(\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}),(\boldsymbol{j}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{j}_h'-\boldsymbol{S}^{-1})\boldsymbol{Q}'\boldsymbol{t}]_{Y'\times Y}. \quad (4.11)$$

On the other hand, we have

$$[\boldsymbol{i}_h'(\boldsymbol{Q}\boldsymbol{S}^{-1}\boldsymbol{F})-\boldsymbol{Q}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{F}_h,\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}]_{X_h'\times X_h}$$
$$=[\boldsymbol{Q}'\boldsymbol{i}_h(\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}),(\boldsymbol{S}^{-1}-\boldsymbol{j}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{j}_h')\boldsymbol{F}]_{Y'\times Y}. \quad (4.12)$$

Hence, from (4.11) and (4.12) we get

$$C\;\|\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}\|_X^2\leqslant[\boldsymbol{Q}'\boldsymbol{i}_h(\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}),(\boldsymbol{j}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{j}_h'-\boldsymbol{S}^{-1})\boldsymbol{Q}'(\boldsymbol{i}_h\boldsymbol{t}_h^*-\boldsymbol{t})]_{Y'\times Y}$$
$$+[\boldsymbol{Q}'\boldsymbol{i}_h(\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}),(\boldsymbol{j}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{j}_h'-\boldsymbol{S}^{-1})(\boldsymbol{Q}'\boldsymbol{t}-\boldsymbol{F})]_{Y'\times Y}. \quad (4.13)$$

From the uniform boundedness of all the operators involved, it is easy to see that

$$|[\boldsymbol{Q}'\boldsymbol{i}_h(\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}),(\boldsymbol{j}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{j}_h'-\boldsymbol{S}^{-1})\boldsymbol{Q}'(\boldsymbol{i}_h\boldsymbol{t}_h^*-\boldsymbol{t})]_{Y'\times Y}|$$
$$\leqslant C\;\|\boldsymbol{t}_h^*-\boldsymbol{t}_{h,\hat{h}}\|_X\,\|\boldsymbol{t}_h^*-\boldsymbol{t}\|_X\;. \quad (4.14)$$

Now, since $(\boldsymbol{\sigma},\xi,u,\eta)=\boldsymbol{S}^{-1}(\boldsymbol{Q}'\boldsymbol{t}-\boldsymbol{F})$, the Strang-type estimate from ([3], Chap. 2, Proposition 2.16) implies in this case that

$$\|(\boldsymbol{j}_h\boldsymbol{S}_{\hat{h}}^{-1}\boldsymbol{j}_h'-\boldsymbol{S}^{-1})(\boldsymbol{Q}'\boldsymbol{t}-\boldsymbol{F})\|_Y\leqslant C\,\{\mathrm{dist}\,((\boldsymbol{\sigma},\xi,u,\eta),\boldsymbol{Y}_h)+\mathrm{dist}\,(\Lambda\xi,H_{\hat{h}}(\Gamma))\},$$

whence

$$
\begin{aligned}
&|\,[\boldsymbol{Q}'\boldsymbol{i}_h(\boldsymbol{t}_h^* - \boldsymbol{t}_{h,\hat{h}}), (\boldsymbol{j}_h \boldsymbol{S}_{\hat{h}}^{-1} \boldsymbol{j}_h' - \boldsymbol{S}^{-1})(\boldsymbol{Q}'\boldsymbol{t} - \boldsymbol{F})]_{Y' \times Y}\,| \\
&\quad \leqslant C \;\|\, \boldsymbol{t}_h^* - \boldsymbol{t}_{h,\hat{h}} \,\|_X \;\; \{\operatorname{dist}((\boldsymbol{\sigma}, \xi, u, \eta), \boldsymbol{Y}_h) + \operatorname{dist}(\Lambda\xi, H_{\hat{h}}(\Gamma))\}\,.
\end{aligned}
\tag{4.15}
$$

Finally, collecting (4.13), (4.8), (4.14) and (4.15) we arrive to

$$
\|\, \boldsymbol{t}_h^* - \boldsymbol{t}_{h,\hat{h}} \,\|_X \leqslant C\,\{\operatorname{dist}(\boldsymbol{t}, \boldsymbol{X}_h) + \operatorname{dist}((\boldsymbol{\sigma}, \xi, u, \eta), \boldsymbol{Y}_h) + \operatorname{dist}(\Lambda\xi, H_{\hat{h}}(\Gamma))\}\,,
$$

which, together with the triangle inequality, (4.8) and (4.7), finishes the proof of the theorem. □

Acknowledgements

This paper was completed while the second author was visiting the Department of Mathematical Sciences of the University of Delaware during the period February 1–11, 1997.

References

[1] G.R. Barrenechea, G.N. Gatica, On the coupling of boundary integral and finite element methods with nonlinear transmission conditions, Appl. Anal. 62 (1996a) 181–210.
[2] G.R. Barrenechea, G.N. Gatica, On the weak solvability of interface problems through mixed finite elements and Dirichlet-to-Neumann methods, Technical Report 96–11, Departamento de Ingeniería Matemática, Universidad de Concepción, J. Math. Anal. Appl. (1996b) submitted.
[3] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, Berlin, 1991.
[4] F. Brezzi, C. Johnson, On the coupling of boundary integral and finite element methods, Calcolo 16 (1979) 189–201.
[5] M. Costabel, Boundary integral operators on Lipschitz domains: elementary results, SIAM J. Math. Anal. 19 (1988) 613–626.
[6] M. Costabel, E.P. Stephan, Coupling of finite and boundary element methods for an elasto-plastic interface problem, SIAM J. Numer. Anal. 27(5) (1990) 1212–1226.
[7] G.N. Gatica, On an alternative variational formulation for the Johnson & Nedelec's coupling procedure, Rev. Mat. Apl. 16 (1995) 17–41.
[8] G.N. Gatica, Combination of mixed finite element and Dirichlet-to-Neumann methods in nonlinear plane elasticity, Appl. Math. Lett. 10 (1997) 29–35.
[9] G.N. Gatica, G.C. Hsiao, The uncoupling of boundary integral and finite element methods for nonlinear boundary value problems. J. Math. Anal. Appl. 189(2) (1995a) 442–461.
[10] G.N. Gatica, G.C. Hsiao, Boundary-Field Equation Methods for a Class of Nonlinear Problems, Pitman Research Notes in Mathematics Series, vol. 331, Longman, New York 1995b.
[11] G.N. Gatica, W.L. Wendland, Coupling of mixed finite elements and boundary elements for linear and nonlinear elliptic problems, Appl. Anal. 63 (1996) 39–75.
[12] G.N. Gatica, W.L. Wendland, Coupling of mixed finite elements and boundary elements for a hyperelastic interface problem, SIAM J. Numer. Anal. 34 (1997) 2335–2356.
[13] G.N. Gatica, W.L. Wendland, Coupling of mixed finite elements and boundary elements in potential theory and elastostatics, Z. Angew. Math. Mech. 75(S2) (1996) 525–526.
[14] G.N. Gatica, W.L. Wendland, On the coupling of mixed finite element and boundary integral methods for a nonlinear elasticity problem, in: L. Morino, W.L. Wendland (Eds.), IABEM Symp. on Boundary Integral Methods for Nonlinear Problems, Kluwer Academic Publishers, Dordrecht, 1997, pp. 75–80.
[15] V. Girault, P.A. Raviart, Finite Element Methods for the Navier-Stokes Equations. Theory and Algorithms, Springer, Berlin, 1986.

[16] H. Han, X. Wu, The approximation of the exact boundary conditions at an artificial boundary for linear elastic equations and its applications, Math. Comput. 59 (1992) 21–37.

[17] G.C. Hsiao, W.L. Wendland, A finite element method for some integral equations of the first kind, J. Math. Anal. Appl. 58 (1977) 449–481.

[18] G.C. Hsiao, S. Zhang, Optimal order multigrid methods for solving exterior boundary value problems, SIAM J. Numer. Anal. 31(3) (1994) 680–694.

[19] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. I, Springer, Berlin, 1972.

[20] J. Necas, Introduction to the Theory of Nonlinear Elliptic Equations, Wiley, New York, 1986.